On the Algebraic Structure of Root-Based Integer Sequences and p-adic Resonance

Oussama Basta

January 20, 2026

Abstract

This paper formalizes a framework for analyzing integer sequences through the lens of fractional roots. We define a transformation function $f(n, r)$ based on the r-th root of an integer n. We demonstrate that this function acts as a "universal translator," allowing mapping between distinct integer sequences via their root powers. Furthermore, we establish a direct link between this framework, the Prime Number Theorem (when $r = n$), and p-adic valuations, proposing that $f(n, r)$ represents a measure of "arithmetic resistance" determined by the prime factorization of n.

1 Introduction and Definitions

Let $n \in \mathbb{Z}^+$ be a positive integer. We consider a root index sequence $r(n)$, which may be a function of n (e.g., $r = n$, $r = n^2$, $r = p_n$). We define the fundamental root difference t as:

$$t(n, r) = n^{1/r} - 1 \tag{1}$$

We observe that for large r, t approaches 0. To analyze the structure of this convergence, we define the reciprocal function $f(n, r)$, which we term the *Arithmetic Resistance*:

$$f(n, r) = \frac{1}{t(n, r)} = \frac{1}{n^{1/r} - 1} \tag{2}$$

2 Asymptotic Behavior and Linearization

To understand the behavior of $f(n, r)$, we apply the Taylor series expansion for the exponential function, noting that $n^{1/r} = \exp\left(\frac{\ln n}{r}\right)$. For sufficiently large r relative to $\ln n$:

$$n^{1/r} \approx 1 + \frac{\ln n}{r} \tag{3}$$

Substituting this approximation into the definition of $f(n, r)$:

$$f(n, r) \approx \frac{1}{\left(1 + \frac{\ln n}{r}\right) - 1} = \frac{r}{\ln n} \tag{4}$$

This linear relationship suggests that $f(n, r)$ scales directly with the root power r and inversely with the natural logarithm of n.

3 The Universal Translation Law

From the definition of f, we can express the integer n in terms of f and r:

$$n^{1/r} = 1 + \frac{1}{f} \implies n = \left(1 + \frac{1}{f}\right)^r \tag{5}$$

Consider two distinct sequences generated by root powers r_1 and r_2, resulting in resistance values f_1 and f_2 for the same integer n. Equating the expressions for n:

$$\left(1+\frac{1}{f_1}\right)^{r_1}=\left(1+\frac{1}{f_2}\right)^{r_2} \tag{6}$$

Solving for f_1 yields the **Closed-Form Translation Law**:

$$f_1(f_2)=\left[\left(1+\frac{1}{f_2}\right)^{r_2/r_1}-1\right]^{-1} \tag{7}$$

Let $K=r_2/r_1$ be the exchange ratio. In the asymptotic limit where f is large, this simplifies via the approximation in Eq. (4):

$$f_1 \approx f_2 \cdot \frac{r_1}{r_2} \tag{8}$$

This implies that any sequence f_i carries the structural information of all other sequences derived from n, convertible solely by the ratio of their root powers.

4 Connection to the Prime Number Theorem

Consider the specific case where the root power is the number itself, i.e., $r(n)=n$. The approximation from Eq. (4) becomes:

$$f(n,n) \approx \frac{n}{\ln n} \tag{9}$$

We recognize the right-hand side as the asymptotic form of the Prime Counting Function, $\pi(n)$, as defined by the Prime Number Theorem:

$$\lim_{n\to\infty}\frac{\pi(n)}{n/\ln n}=1 \implies f(n,n)\sim\pi(n) \tag{10}$$

Thus, the arithmetic resistance of an integer against its own root is asymptotically equivalent to the density of primes up to that integer.

5 The p-adic Formulation

The structure of $f(n,r)$ can be further decomposed by the Fundamental Theorem of Arithmetic. Let the prime factorization of n be:

$$n=\prod_p p^{v_p(n)} \tag{11}$$

where $v_p(n)$ is the p-adic valuation of n (the exponent of prime p in n). Substituting this into the logarithmic approximation:

$$\ln n=\sum_p v_p(n)\ln p \tag{12}$$

Therefore, the function $f(n,r)$ can be expressed in terms of p-adic valuations:

$$f(n,r)\approx\frac{r}{\sum_p v_p(n)\ln p} \tag{13}$$

5.1 Interpretation

This formulation reveals the microscopic behavior of the sequence:

- **Primes:** If n is prime ($n=p$), then $v_p(n)=1$ and all other valuations are 0. The denominator is small ($\ln p$), maximizing $f(n,r)$.
- **Composites:** If n is highly composite, the sum $\sum v_p(n)\ln p$ is large, minimizing $f(n,r)$.

This explains the observed "spikes" in the sequence data: $f(n,r)$ acts as a detector for prime numbers, reacting to the low "resistance" (logarithmic sum) of primes compared to composite numbers.